II.C.2.a. Derivation of kinematics equations

Let $\vec{V}$ represent the velocity of the center-of-mass (COM) system relative to the laboratory system. Before the interaction, the relationships between the velocities are

$$\frac{\vec{p}}{m} = \frac{\vec{K}}{m} + \vec{V} \quad \text{and} \quad 0 = -\frac{\vec{K}}{M} + \vec{V} \quad , \tag{II C2 a.1}$$

from which we can solve for $\vec{V}$ and $\vec{K}$ in terms of $\vec{p}$:

$$\vec{V} = \frac{\vec{K}}{M} \quad \text{and} \quad \vec{\mathrm{K}} = \frac{\mathrm{M}}{\mathrm{m+M}}\vec{p} \;, \quad \text{which implies} \quad \vec{V} = \frac{\vec{p}}{(m+M)} \quad . \tag{II C2 a.2}$$

The total energy in the lab must equal the energy *in* the COM plus the energy *of* the COM. Before the interaction, this gives us

$$\begin{aligned} E_{\text{lab}} &= E_{\text{of COM}} + E_{\text{in COM}} \\ \frac{p^2}{2m} + m + M &= \frac{(m+M)V^2}{2} + \frac{K^2}{2m} + \frac{K^2}{2M} + m + M \quad , \end{aligned} \tag{II C2 a.3}$$

which is clearly true, as can be seen by substitution of the expressions in Eq. (II C2 a.2) into (II C2 a.3). We are using non-relativistic energies but nevertheless including the masses because they may be different before and after the interaction. Within the COM, conservation of energy requires that the initial and final energies are equal:

$$\begin{aligned} E_{\text{in com}} &= E'_{\text{in com}} \\ \frac{K^2}{2m} + \frac{K^2}{2M} + m + M &= \frac{K'^2}{2m'} + \frac{K'^2}{2M'} + m' + M' \quad . \end{aligned} \tag{II C2 a.4}$$

Solving for K' in terms of K gives

$$\begin{aligned} \frac{K'^2}{2}\left(\frac{1}{m'} + \frac{1}{M'}\right) &= \frac{K^2}{2}\left(\frac{1}{m} + \frac{1}{M}\right) + \left(m + M - (m' + M')\right) \quad \text{or} \\ \frac{K'^2}{2}\left(\frac{m' + M'}{m'M'}\right) &= \frac{K^2}{2}\left(\frac{m+M}{mM}\right) + (Q) \quad , \end{aligned} \tag{II C2 a.5}$$

in which we have defined the Q-value as

$$Q = m + M - m' - M' \quad . \tag{II C2 a.6}$$

Rewriting Eq. (II C2 a.5) using the value for K from Eq. (II C2 a.2) gives

$$\begin{aligned} K'^2 &= \frac{2m'M'}{m'+M'}\left[K^2\left(\frac{m+M}{2mM}\right)+Q\right] \\ &= \frac{2m'M'}{m'+M'}\left[\left\{p\frac{M}{m+M}\right\}^2\left(\frac{m+M}{2mM}\right)+Q\right] \\ &= \frac{2m'M'}{m'+M'}\left[\frac{p^2}{2m}\frac{M}{(m+M)}+Q\right] \quad . \end{aligned} \tag{II C2 a.7}$$

This can also be written as

$$K'^2 = \frac{2m'M'}{m'+M'}\left[E\frac{M}{(m+M)}+Q\right] \quad , \tag{II C2 a.8}$$

in which E is equal to the kinetic energy of the incident particle in the laboratory system,

$$E = \frac{p^2}{2m} \quad . \tag{II C2 a.9}$$

This definition of E is used throughout this manual; cross sections are always specified in terms of this energy unless otherwise noted explicitly.

The transformation from COM to laboratory gives values for momenta after the interaction. Again, we add velocities, similar to Eq. (II C2 a.1), using Eq. (II C2 a.2) for $\vec{V}$:

$$\frac{\vec{p}\,'}{m'} = \frac{\vec{K}'}{m'} + \vec{V} = \frac{\vec{K}'}{m'} + \frac{\vec{p}}{(m+M)} \quad . \tag{II C2 a.10}$$

(An analogous set of equations holds for the second particle,

$$\frac{\vec{q}\,'}{M'} = -\frac{\vec{K}}{M} + \vec{V} = -\frac{\vec{K}}{M} + \frac{\vec{p}}{(m+M)} \quad , \tag{II C2 a.11}$$

but we shall not be concerned with this particle now.)

Setting $\mu = \cos\theta$ and $\nu = \cos\beta$, we can write Eq. (II C2 a.10) in terms of components

$$\frac{p'\cos\theta}{m'} = \frac{K'\cos\beta}{m'} + \frac{p}{(m+M)} \quad \text{and} \quad \frac{p'\sin q}{m'} = \frac{K'\sin\beta}{m'} + 0 \qquad \text{(II C2 a.12)}$$

or

$$p'\mu = K'\nu + \frac{m'p}{(m+M)} \quad \text{and} \quad p'\sqrt{1-\mu^2} = K'\sqrt{1-\nu^2} \quad , \qquad \text{(II C2 a.13)}$$

in which we have set $\mu = \cos\theta$ and $\nu = \cos\beta$. Squaring and adding the two equations in (II C2 a.13) gives

$$\left(\frac{p'\mu}{m'}\right)^2 + \left(\frac{p'\sqrt{1-\mu^2}}{m'}\right)^2 = \left(\frac{K'\nu}{m'}\right)^2 + \left(\frac{K'\sqrt{1-\nu^2}}{m'}\right)^2 + 2\frac{K'\nu}{m'}\frac{p}{(m+M)} + \left(\frac{p}{(m+M)}\right)^2, \qquad \text{(II C2 a.14)}$$

or

$$\frac{p'^2}{m'^2} = \frac{K'^2}{m'^2} + \frac{2K'\nu p}{m'(m+M)} + \frac{p^2}{(m+M)^2} \quad . \qquad \text{(II C2 a.15)}$$

Replacing $K'\nu$ with its equivalent from Eq. (II C2 a.13) puts Eq. (II C2 a.15) into the form

$$\frac{p'^2}{m'^2} = \frac{K'^2}{m'^2} + \frac{2p}{m'(m+M)}\left\{p'\mu - \frac{m'p}{m+M}\right\} + \frac{p^2}{(m+M)^2} \quad , \qquad \text{(II C2 a.16)}$$

which can be rearranged as

$$p'^2 = K'^2 + \frac{2m'p\mu}{(m+M)}p' - \frac{m'^2p^2}{(m+M)^2} \quad . \qquad \text{(II C2 a.17)}$$

Solving for p' in terms of other quantities gives

$$\begin{aligned} p' &= \frac{m'}{m+M}p\mu + \sqrt{\left(\frac{m'}{m+M}\right)^2 p^2\mu^2 - \left(\frac{m'}{m+M}\right)^2 p^2 + K'^2} \\ &= \frac{m'}{m+M}\left\{p\mu + \sqrt{p^2\left(\mu^2-1\right) + \left(\frac{m+M}{m'}\right)^2 K'^2}\right\} \quad . \end{aligned} \qquad \text{(II C2 a.18)}$$

(Consideration of the $p = 0$ limit confirms that this choice of sign for the radical is appropriate.)

From Eq. (II C2 a.7), we know K' in terms of p. Therefore, to simplify Eq. (II C2 a.18), we define ξ as

$$\xi = \left(\frac{m'}{m+M} \right) \frac{p}{K'} \quad . \qquad \text{(II C2 a.19)}$$

Using this definition of ξ, Eq. (II C2 a.18) can be put into the form

$$\begin{aligned} p' &= \frac{m'}{m+M} \left\{ p\mu + \sqrt{p^2\left(\mu^2-1\right)+\xi^{-2}p^2} \right\} \\ &= \frac{m'}{m+M}\frac{p}{\xi} \left\{ \xi\,\mu + \sqrt{1-\xi^2\left(1-\mu^2\right)} \right\} \quad . \end{aligned} \qquad \text{(II C2 a.20)}$$

The quantity outside the curly brackets is exactly equal to K'; making this substitution gives

$$p' = K'\left\{ \xi\,\mu + \sqrt{1-\xi^2\left(1-\mu^2\right)} \right\} \quad . \qquad \text{(II C2 a.21)}$$

The laboratory energy of the outgoing particle can then be found as

$$E' = \frac{p'^2}{2m'} = \frac{K'^2}{2m'} \left\{ \xi\,\mu + \sqrt{1-\xi^2\left(1-\mu^2\right)} \right\}^2 \quad , \qquad \text{(II C2 a.22)}$$

or, using Eq. (II C2 a.8) for K',

$$E' = \frac{M'}{m'+M'} \left\{ \xi\,\mu + \sqrt{1-\xi^2\left(1-\mu^2\right)} \right\}^2 \left[E\frac{M}{(m+M)} + Q \right] \quad . \qquad \text{(II C2 a.23)}$$

It is customary to define the laboratory threshold energy, here denoted by Ξ, as

$$\Xi \equiv -\frac{m+M}{M}Q \quad . \qquad \text{(II C2 a.24)}$$

In terms of Ξ, Eq. (II C2 a.23) for E' becomes

$$E' = \frac{M'}{m'+M'}\frac{M}{(m+M)} \left\{ \xi\,\mu + \sqrt{1-\xi^2\left(1-\mu^2\right)} \right\}^2 \left[E - \Xi \right] \quad . \qquad \text{(II C2 a.25)}$$

Equation (II C2 a.19) for ξ can also be written in terms of Ξ, using Eq. (II C2 a.8), as

$$\xi^2 = \frac{\left(\frac{m'}{m+M}\right)^2 2mE}{\frac{2m'M'}{m'+M'}\left[E\frac{M}{(m+M)}+Q\right]} = \frac{m'}{M'}\frac{m}{M}\frac{m'+M'}{m+M}\frac{E}{E-\Xi} \quad . \qquad \text{(II C2 a.26)}$$

Next, we consider the transformation of angle from laboratory θ to COM β and vice versa. From Eq. (II C2 a.13) we have

$$p'\mu = K'\nu + \frac{m'p}{(m+M)} = K'\nu + K'\xi \quad , \qquad \text{(II C2 a.27)}$$

in which we have made use of Eq. (II C2 a.19). Substituting Eq. (II C2 a.21) into this equation gives

$$p'\mu = K'\left\{\xi\mu + \sqrt{1-\xi^2\left(1-\mu^2\right)}\right\}\mu = K'\nu + K'\xi \quad , \qquad \text{(II C2 a.28)}$$

which reduces to

$$\nu = -\xi\left(1-\mu^2\right) + \mu\sqrt{1-\xi^2\left(1-\mu^2\right)} \quad . \qquad \text{(II C2 a.29)}$$

This equation can be inverted to give μ in terms of ν as follows:

$$\begin{gathered}\left[\nu+\xi\left(1-\mu^2\right)\right]^2 = \mu^2\left[1-\xi^2\left(1-\mu^2\right)\right] \quad , \\ \nu^2 + 2\xi\nu\left(1-\mu^2\right) + \xi^2\left(1-2\mu^2+\mu^4\right) = \mu^2 - \xi^2\mu^2 + \xi^2\mu^4 \quad , \\ \mu^2\left(1+\xi^2+2\xi\nu\right) = \nu^2 + 2\xi\nu + \xi^2 \quad , \\ \mu^2 = \frac{\nu^2+2\xi\nu+\xi^2}{\left(1+\xi^2+2\xi\nu\right)} \quad ,\end{gathered} \qquad \text{(II C2 a.30)}$$

or, finally, as

$$\mu = \frac{\nu+\xi}{\sqrt{1+\xi^2+2\xi\nu}} \quad . \qquad \text{(II C2 a.31)}$$

The transformation of cross section from COM to lab requires the derivative of ν with respect to μ; this is found from Eq. (II C2 a.29):

$$\begin{aligned}\frac{d\nu}{d\mu} &= \frac{d}{d\mu}\left\{-\xi\left(1-\mu^2\right)+\mu\sqrt{1-\xi^2\left(1-\mu^2\right)}\right\} \\ &= 2\mu\xi+\sqrt{1-\xi^2\left(1-\mu^2\right)}+\frac{\mu\, 2\mu\xi^2\,\frac{1}{2}}{\sqrt{1-\xi^2\left(1-\mu^2\right)}} \\ &= \frac{2\mu\xi\sqrt{1-\xi^2\left(1-\mu^2\right)}+1-\xi^2\left(1-\mu^2\right)+\mu^2\xi^2}{\sqrt{1-\xi^2\left(1-\mu^2\right)}} \quad ,\end{aligned} \tag{II C2 a.32}$$

giving, finally, the expression for the derivative

$$\frac{d\nu}{d\mu}=\frac{\left(\mu\xi+\sqrt{1-\xi^2\left(1-\mu^2\right)}\right)^2}{\sqrt{1-\xi^2\left(1-\mu^2\right)}} \quad . \tag{II C2 a.33}$$