VIII.D.1. Relation of SAMMY/URR Radius to ENDF/B Radius

To understand the rationale for the choice of $R\,'$ as the input radius for ENDF, we consider the value of the cross section in the limit in which the effect of resonances is negligible. From Eq. (VIII A.2), the average scattering matrix is given by

$$\left\langle S_{cc} \right\rangle = e^{-2i\varphi_c} \; \frac{1-\left\langle R_{cc} \right\rangle L_c^{0*}}{1-\left\langle R_{cc} \right\rangle L_c^{0}} \quad , \tag{VIII D1.1}$$

in which the average R-matrix is

$$\left\langle R_{cc} \right\rangle = R_c^{\infty} + \; i\,\pi\, s_c \quad . \tag{VIII D1.2}$$

Substituting iP for L, dropping subscripts for simplicity, and rearranging give

$$\begin{aligned}
\left\langle S_{cc} \right\rangle &= e^{-2i\varphi} \frac{\left[1-\left(R^{\infty}+i\,\pi\, s\right)\left(-i\,P\right)\right]}{\left[1-\left(R^{\infty}+i\,\pi\, s\right)\left(+i\,P\right)\right]} \;=\; e^{-2i\varphi}\frac{\left[1-\pi Ps+iPR^{\infty}\right]}{\left[1+\pi Ps-iPR^{\infty}\right]} \\
&= e^{-2i\varphi}\frac{\left[1-\pi Ps+iPR^{\infty}\right]\left[1+\pi Ps+iPR^{\infty}\right]}{\left[1+\pi Ps-iPR^{\infty}\right]\left[1+\pi Ps+iPR^{\infty}\right]} \\
&= \left[\cos\left(2\varphi\right)-i\sin\left(2\varphi\right)\right]\frac{1-\pi^2P^2s^2-\left(PR^{\infty}\right)^2+2iPR^{\infty}}{\left(1+\pi Ps\right)^2+\left(PR^{\infty}\right)^2} \\
&= \frac{\begin{aligned}&\Big[\Big\{\cos\left(2\varphi\right)\left[1-\pi^2P^2s^2-\left(PR^{\infty}\right)^2\right]+\sin\left(2\varphi\right)2PR^{\infty}\Big\} \\ &\quad +i\Big\{-\sin\left(2\varphi\right)\left[1-\pi^2P^2s^2-\left(PR^{\infty}\right)^2\right]+\sin\left(2\varphi\right)2PR^{\infty}\Big\}\Big]\end{aligned}}{\left(1+\pi Ps\right)^2+\left(PR^{\infty}\right)^2} \quad .
\end{aligned} \tag{VIII D1.3}$$

The cross section can then be expressed as

$$\begin{aligned}
\left\langle \sigma_c \right\rangle &= \frac{2\pi\, g}{k^2}\left(1-\mathrm{Re}\left\langle S_{cc}\right\rangle\right) \\
&= \frac{2\pi\, g}{k^2}\left(1-\frac{\cos\left(2\varphi\right)\left[1-\pi^2P^2s^2-\left(PR^{\infty}\right)^2\right]+\sin\left(2\varphi\right)2PR^{\infty}}{\left(1+\pi Ps\right)^2+\left(PR^{\infty}\right)^2}\right) \quad .
\end{aligned} \tag{VIII D1.4}$$

If we consider only s-waves ($l = 0$), both P and φ have the value $\rho = ka$. The expression for the cross section then becomes

$$\begin{aligned}
\langle\sigma_c\rangle &\approx \frac{2\pi g}{k^2}\left(1-\frac{\left(1-2\rho^2\right)\left[1-\pi^2\rho^2 s^2-\rho^2\left(R^\infty\right)^2\right]+4\rho^2 R^\infty}{\left(1+\pi\rho s\right)^2+\rho^2\left(R^\infty\right)^2}\right) \\
&= \frac{2\pi g}{k^2}\,\frac{\left(1+\pi\rho s\right)^2+\rho^2\left(R^\infty\right)^2-\left(1-2\rho^2\right)\left[1-\pi^2\rho^2 s^2-\rho^2\left(R^\infty\right)^2\right]-4\rho^2 R^\infty}{\left(1+\pi\rho s\right)^2+\rho^2\left(R^\infty\right)^2} \\
&\approx \frac{2\pi g}{k^2}\,\frac{1+2\pi\rho s+\pi^2\rho^2 s^2+\rho^2\left(R^\infty\right)^2-1+\pi^2\rho^2 s^2+\rho^2\left(R^\infty\right)^2+2\rho^2-4\rho^2 R^\infty}{\left(1+\pi\rho s\right)^2+\rho^2\left(R^\infty\right)^2}\quad ,
\end{aligned} \tag{VIII D1.5}$$

in which cos and sin have been expanded for small values of ρ, and terms $\sim\rho^4$ have been dropped in the numerator. Simplifying, we find

$$\begin{aligned}
\langle\sigma_c\rangle &\approx \frac{2\pi g}{k^2}\,\frac{\left[2\pi\rho s+2\pi^2\rho^2 s^2\right]+2\rho^2\left(R^\infty\right)^2+2\rho^2-4\rho^2 R^\infty}{\left(1+\pi\rho s\right)^2+\rho^2\left(R^\infty\right)^2} \\
&= \frac{4\pi g}{k^2}\,\frac{\pi\rho s+\pi^2\rho^2 s^2+\rho^2\left[1-\left(R^\infty\right)\right]^2}{\left(1+\pi\rho s\right)^2+\rho^2\left(R^\infty\right)^2}\quad .
\end{aligned} \tag{VIII D1.6}$$

In the limit in which all the resonances are considered to be distant resonances, the pole strength s is zero and the cross section becomes

$$\langle\sigma_c\rangle \approx \frac{4\pi g}{k^2}\,\frac{\rho^2\left[1-\left(R^\infty\right)\right]^2}{1+\rho^2\left(R^\infty\right)^2} \approx \frac{4\pi g}{k^2}k^2a^2\left[1-\left(R^\infty\right)\right]^2 = 4\pi g R'^2 \quad , \tag{VIII D1.7}$$

in which R' is defined as $a\left(1-R^\infty\right)$. This is the customary low-energy formula, even in the resolved resonance region.

We now consider the more general case, for all energies. In Eq. (VIII D1.4), assume that the pole strength s is negligible but do not use the low-energy limit. The cross section can be shown to have the form

$$\langle \sigma_c \rangle \approx \frac{2\pi g}{k^2}\left(1-\frac{\cos(2\varphi)\left[1-\left(PR^\infty\right)^2\right]+\sin(2\varphi)2PR^\infty}{1+\left(PR^\infty\right)^2}\right)$$

$$= \frac{2\pi g}{k^2}\sin^2(\varphi-\varepsilon) \quad , \tag{VIII D1.8}$$

where the phase shift ε is given by

$$\varepsilon = \tan^{-1}\left(P\,R^\infty\right) \quad . \tag{VIII D1.9}$$

To see this, note that

$$\begin{aligned}\sin^2(\varphi-\varepsilon) &= \left(\sin\varphi\,\cos\varepsilon-\cos\varphi\,\sin\varepsilon\right)^2\\ &= \sin^2\varphi\,\cos^2\varepsilon-2\sin\varphi\,\cos\varphi\,\cos\varepsilon\,\sin\varepsilon+\cos^2\varphi\,\sin^2\varepsilon\\ &= \tfrac{1}{2}\left[1-\cos(2\varphi)\right]\cos^2\varepsilon-2\left[\sin(2\varphi)/2\right]\cos\varepsilon\,\sin\varepsilon+\tfrac{1}{2}\left[1+\cos(2\varphi)\right]\sin^2\varepsilon\\ &= \tfrac{1}{2}\left\{1+\cos(2\varphi)\left[\sin^2\varepsilon-\cos^2\varepsilon\right]-\sin(2\varphi)\sin\varepsilon\,\cos\varepsilon\right\} \quad .\end{aligned} \tag{VIII D1.10}$$

From Eq. (VIII D1.9) it can readily be seen that

$$\begin{aligned}\tan\varepsilon &= PR^\infty\\ \sin\varepsilon &= \frac{PR^\infty}{\sqrt{1+\left(PR^\infty\right)^2}}\\ \cos\varepsilon &= \frac{1}{\sqrt{1+\left(PR^\infty\right)^2}}\\ \sin\varepsilon\,\cos\varepsilon &= \frac{PR^\infty}{1+\left(PR^\infty\right)^2}\\ \sin^2\varepsilon-\cos^2\varepsilon &= -\frac{1-\left(PR^\infty\right)^2}{1+\left(PR^\infty\right)^2} \quad .\end{aligned} \tag{VIII D1.11}$$

Substituting these values into Eq. (VIII D1.10) reproduces (VIII D1.8) exactly.