II.D.3. Details Involving Derivatives

In this section are presented some of the algebraic details relating to the partial derivatives. We first consider the derivative of one complex variable with respect to another, and then the derivative of the inverse of a matrix quantity. Both are needed in Section II.D.1.a.

Derivative of one complex variable with respect to another

Given any two complex variables $A = A^r + iA^i$ and $B = B^r + iB^i$, where A is an analytical function of B, the derivative of the components of A with respect to the components of B may be expressed as follows:

$$\frac{\partial A^r}{\partial B^r} = \mathrm{Re}\frac{\partial A}{\partial B^r} = \mathrm{Re}\frac{\partial A}{\partial B}\frac{\partial B}{\partial B^r} = \mathrm{Re}\frac{\partial A}{\partial B} \quad , \tag{II D3.1}$$

$$\frac{\partial A^r}{\partial B^i} = \mathrm{Re}\frac{\partial A}{\partial B^i} = \mathrm{Re}\frac{\partial A}{\partial B}\frac{\partial B}{\partial B^i} = \mathrm{Re}\frac{\partial A}{\partial B}(i) = -\mathrm{Im}\frac{\partial A}{\partial B} \quad , \tag{II D3.2}$$

$$\frac{\partial A^i}{\partial B^r} = \mathrm{Im}\frac{\partial A}{\partial B^r} = \mathrm{Im}\frac{\partial A}{\partial B}\frac{\partial B}{\partial B^r} = \mathrm{Im}\frac{\partial A}{\partial B} \quad , \tag{II D3.3}$$

and

$$\frac{\partial A^i}{\partial B^i} = \mathrm{Im}\frac{\partial A}{\partial B^i} = \mathrm{Im}\frac{\partial A}{\partial B}\frac{\partial B}{\partial B^i} = \mathrm{Im}\frac{\partial A}{\partial B}(i) = \mathrm{Re}\frac{\partial A}{\partial B} \quad . \tag{II D3.4}$$

Also, the usual chain rule applies:

$$\begin{aligned}\frac{\partial A^r}{\partial C^r} &= \frac{\partial A^r}{\partial B^r}\frac{\partial B^r}{\partial C^r} + \frac{\partial A^r}{\partial B^i}\frac{\partial B^i}{\partial C^r} \\ &= \left[\mathrm{Re}\left(\frac{\partial A}{\partial B}\right)\right]\left[\mathrm{Re}\left(\frac{\partial B}{\partial C}\right)\right] + \left[-\mathrm{Im}\left(\frac{\partial A}{\partial B}\right)\right]\left[\mathrm{Im}\left(\frac{\partial B}{\partial C}\right)\right] = \mathrm{Re}\left[\frac{\partial A}{\partial B}\frac{\partial B}{\partial C}\right] \quad ,\end{aligned} \tag{II D3.5}$$

$$\begin{aligned}\frac{\partial A^r}{\partial C^i} &= \frac{\partial A^r}{\partial B^r}\frac{\partial B^r}{\partial C^i} + \frac{\partial A^r}{\partial B^i}\frac{\partial B^i}{\partial C^i} \\ &= \left[\mathrm{Re}\left(\frac{\partial A}{\partial B}\right)\right]\left[-\mathrm{Im}\left(\frac{\partial B}{\partial C}\right)\right] + \left[-\mathrm{Im}\left(\frac{\partial A}{\partial B}\right)\right]\left[\mathrm{Re}\left(\frac{\partial B}{\partial C}\right)\right] = -\mathrm{Im}\left[\frac{\partial A}{\partial B}\frac{\partial B}{\partial C}\right] \quad ,\end{aligned} \tag{II D3.6}$$

$$\frac{\partial A^i}{\partial C^r} = \frac{\partial A^i}{\partial B^r}\frac{\partial B^r}{\partial C^r} + \frac{\partial A^i}{\partial B^i}\frac{\partial B^i}{\partial C^r}$$
$$= \left[\mathrm{Im}\left(\frac{\partial A}{\partial B}\right)\right]\left[\mathrm{Re}\left(\frac{\partial B}{\partial C}\right)\right] + \left[\mathrm{Re}\left(\frac{\partial A}{\partial B}\right)\right]\left[\mathrm{Im}\left(\frac{\partial B}{\partial C}\right)\right] = \mathrm{Im}\left[\frac{\partial A}{\partial B}\frac{\partial B}{\partial C}\right] \quad , \qquad \text{(II D3.7)}$$

and

$$\frac{\partial A^i}{\partial C^i} = \frac{\partial A^i}{\partial B^r}\frac{\partial B^r}{\partial C^i} + \frac{\partial A^i}{\partial B^i}\frac{\partial B^i}{\partial C^i}$$
$$= \left[\mathrm{Im}\left(\frac{\partial A}{\partial B}\right)\right]\left[-\mathrm{Im}\left(\frac{\partial B}{\partial C}\right)\right] + \left[\mathrm{Re}\left(\frac{\partial A}{\partial B}\right)\right]\left[\mathrm{Re}\left(\frac{\partial B}{\partial C}\right)\right] = \mathrm{Re}\left[\frac{\partial A}{\partial B}\frac{\partial B}{\partial C}\right] \quad . \qquad \text{(II D3.8)}$$

Derivative of the inverse of a matrix

In Eq. (II D1 a.13), the quantity Y is defined as

$$Y_{ef} = \left[\left(L^{-1} - R\right)^{-1}\right]_{ef} \quad . \qquad \text{(II D3.9)}$$

To find the derivative of Y with respect to R, we first note that

$$\sum_a Y_{ea} Y_{ab}^{-1} = \delta_{eb} \quad , \qquad \text{(II D3.10)}$$

so that the derivative is zero; that is,

$$0 = \frac{\partial}{\partial R_{cd}}\left[\sum_a Y_{ea} Y_{ab}^{-1}\right] = \sum_a \frac{\partial Y_{ea}}{\partial R_{cd}} Y_{ab}^{-1} + \sum_a Y_{ea}\frac{\partial Y_{ab}^{-1}}{\partial R_{cd}}$$
$$= \sum_a \frac{\partial Y_{ea}}{\partial R_{cd}} Y_{ab}^{-1} + \sum_a Y_{ea}\left\{-\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\left(1-\delta_{cd}\right)\right\} \quad . \qquad \text{(II D3.11)}$$

The quantity in curly brackets comes from the symmetry of the R-matrix and from the stipulation that only unique matrix elements are to be considered [$c \le d,\ e \le f$ in Eq. (II D1.1) and Eq. (II D1 a.1)]. Multiplying both terms by Y_{bf}, summing over b, and rearranging give

$$\sum_a \frac{\partial Y_{ea}}{\partial R_{cd}} \sum_b Y_{ab}^{-1} Y_{bf} \;=\; \sum_{a,b} Y_{ea} \left\{ \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\left(1-\delta_{cd}\right) \right\} Y_{bf} \quad ,$$

$$\sum_a \frac{\partial Y_{ea}}{\partial R_{cd}} \delta_{af} = Y_{ec}Y_{df} + Y_{ed}Y_{cf}\left(1-\delta_{cd}\right) \quad , \qquad \text{(II D3.12)}$$

or, finally,

$$\frac{\partial Y_{ef}}{\partial R_{cd}} = Y_{ec}Y_{df} + Y_{ed}Y_{cf}\left(1-\delta_{cd}\right) \quad . \qquad \text{(II D3.13)}$$

This is the derivative used in Eq. (II D1 a.14).