II.A.1. R-Matrix and A-Matrix Equations

The R-matrix was introduced in Eq. (II A.6) as

$$W = P^{1/2}(I - RL)^{-1}(I - RL^{*})P^{-1/2} \quad , \tag{II A1.1}$$

but the formula for the R-matrix was not given there. If λ represents a particular resonance (or level), then the general form for the R-matrix is

$$R_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c}\, \gamma_{\lambda c'}}{E_{\lambda} - E}\, \delta_{JJ'} \quad , \tag{II A1.2}$$

where E_{λ} represents the energy of the resonance, and the reduced width amplitude γ is related to the partial width Γ by

$$\Gamma_{\lambda \boldsymbol{c}} = 2\, \boldsymbol{P_c}\, \gamma_{\lambda \boldsymbol{c}}^{2} \quad . \tag{II A1.3}$$

The sum in Eq. (II A1.2) contains an infinite number of levels. All channels, including the "gamma channel" for which one of the particles is a photon, are represented by the channel indices.

The R-matrix is not the only possibility for parameterization of the scattering matrix. In the R-matrix formulation, equations are expressed in terms of channel-channel interactions. It is also possible to formulate scattering theory in terms of level-level interactions; this formulation uses what is called the A-matrix, which is defined as

$$A_{\mu\lambda}^{-1} = \left(E_{\lambda} - E\right)\delta_{\mu\lambda} - \sum_{c} \gamma_{\mu c}\, L_c\, \gamma_{\lambda c} \quad . \tag{II A1.4}$$

To see the relationship of the A-matrix to the R-matrix, we begin by multiplying both sides of Eq. (II A1.4) by A and summing over λ:

$$\sum_{\lambda} A_{\mu\lambda}^{-1}\, A_{\lambda\nu} = \sum_{\lambda} \left(E_{\lambda} - E\right)\delta_{\mu\lambda}\, A_{\lambda\nu} - \sum_{c} \gamma_{\mu c}\, L_c \sum_{\lambda} \gamma_{\lambda c}\, A_{\lambda\nu} \quad ,$$

or

$$\delta_{\mu\nu} = \left(E_{\mu} - E\right) A_{\mu\nu} - \sum_{c} \gamma_{\mu c}\, L_c \sum_{\lambda} \gamma_{\lambda c}\, A_{\lambda\nu} \quad . \tag{II A1.5}$$

Dividing by $\left(E_{\mu} - E\right)$, multiplying on the left by $\gamma_{\mu c'}$ and on the right by $\gamma_{\nu c''}$, and summing over μ puts this equation into the form

$$\begin{aligned} \sum_{\mu} \gamma_{\mu c'} \left(E_{\mu} - E\right)^{-1} \delta_{\mu\nu}\, \gamma_{\nu c''} &= \sum_{\mu} \gamma_{\mu c'} \left(E_{\mu} - E\right)^{-1} \left(E_{\mu} - E\right) A_{\mu\nu}\, \gamma_{\nu c''} \\ &\quad - \sum_{\mu} \gamma_{\mu c'} \left(E_{\mu} - E\right)^{-1} \sum_{c} \gamma_{\mu c}\, L_c \sum_{\lambda} \gamma_{\lambda c} A_{\lambda\nu}\, \gamma_{\nu c''} \quad , \end{aligned} \tag{II A1.6}$$

which can be reduced to

$$\gamma_{\nu c'} \left(E_\nu - E\right)^{-1} \gamma_{\nu c"} = \sum_{\mu} \gamma_{\mu c'} A_{\mu\nu} \gamma_{\nu c"} - \sum_{c} \left[\sum_{\mu} \gamma_{\mu c'} \left(E_\mu - E\right)^{-1} \gamma_{\mu c} \right] L_c \sum_{\lambda} \gamma_{\lambda c} A_{\lambda\nu} \gamma_{\nu c"} \quad . \tag{II A1.7}$$

Summing over ν puts this into the form

$$\left[\sum_{\nu} \gamma_{\nu c'} \left(E_\nu - E\right)^{-1} \gamma_{\nu c"} \right] = \sum_{\mu\nu} \gamma_{\mu c'} A_{\mu\nu} \gamma_{\nu c"} - \sum_{c} \left[\sum_{\mu} \gamma_{\mu c'} \left(E_\mu - E\right)^{-1} \gamma_{\mu c} \right] L_c \sum_{\lambda\nu} \gamma_{\lambda c} A_{\lambda\nu} \gamma_{\nu c"} \quad , \tag{II A1.8}$$

in which we can replace the quantities in square brackets by the R-matrix, giving

$$\begin{aligned} R_{c'c"} &= \sum_{\mu\nu} \gamma_{\mu c'} A_{\mu\nu} \gamma_{\nu c"} - \sum_{c} R_{c'c} L_c \sum_{\lambda\nu} \gamma_{\lambda c} A_{\lambda\nu} \gamma_{\nu c"} \\ &= \sum_{c} \left[\delta_{c'c} - R_{c'c} L_c \right] \sum_{\lambda\nu} \gamma_{\lambda c} A_{\lambda\nu} \gamma_{\nu c"} \quad . \end{aligned} \tag{II A1.9}$$

Solving for the summation, this equation can be rewritten as

$$\left[\left(I - RL\right)^{-1} R \right]_{cc"} = \sum_{\lambda\nu} \gamma_{\lambda c} A_{\lambda\nu} \gamma_{\nu c"} \quad . \tag{II A1.10}$$

To relate this to the scattering matrix, we note that Eq. (II A.6) can be rewritten using Eq. (II A.7) into the form

$$\begin{aligned} W &= P^{1/2} \left(I - RL\right)^{-1} \left(I - RL^*\right) P^{-1/2} \\ &= P^{1/2} \left(I - RL\right)^{-1} \left(I - RL + 2iRP\right) P^{-1/2} \\ &= P^{1/2} \left[\left(I - RL\right)^{-1} \left(I - RL\right) + 2i \left(I - RL\right)^{-1} RP \right] P^{-1/2} \\ &= P^{1/2} P^{-1/2} + 2iP^{1/2} \left(I - RL\right)^{-1} RPP^{-1/2} \\ &= I + 2iP^{1/2} \left(I - RL\right)^{-1} RP^{1/2} \quad . \end{aligned} \tag{II A1.11}$$

Comparing Eq. (II A1.10) to Eq. (II A1.11) gives, in matrix form,

$$W = I + 2iP^{1/2} \gamma A \gamma P^{1/2} \quad . \tag{II A1.12}$$

These equations are exact; no approximations have been made.

One common approximation should be discussed here: the "eliminated channel" approximation, for which one particular type of channel is treated in aggregate and assumed to not interfere from level to level. This is most easily understood in the A-matrix definition, Eq. (II A1.4); assuming no level-level interference for the gamma channels (for example), this equation can be approximated as

$$A_{\mu\lambda}^{-1} \approx \left(E_\lambda - E\right)\delta_{\mu\lambda} - \left[\sum_{\substack{\gamma=\text{gamma}\\ \text{channels}}} \gamma_{\mu\gamma}\, L_\gamma\, \gamma_{\lambda\gamma}\right]\delta_{\mu\lambda} - \sum_{\substack{c=\text{particle}\\ \text{channels}}} \gamma_{\mu c}\, L_c\, \gamma_{\lambda c} \quad . \tag{II A1.13}$$

The quantity in square brackets corresponds to those channels for which the level-level interference is to be neglected; that is, only the interactions within one level are important. For gamma channels, $L = S+iP$ reduces to $L= i$, so Eq. (II A1.13) becomes

$$A_{\mu\lambda}^{-1} \approx \left(E_\lambda - E - i\overline{\Gamma}_{\lambda\gamma}/2\right)\delta_{\mu\lambda} - \sum_{\substack{c=\text{particle}\\ \text{channels}}} \gamma_{\mu c}\, L_c\, \gamma_{\lambda c} \quad . \tag{II A1.14}$$

The bar over $\overline{\Gamma}_{\lambda\gamma}$ is used to indicate the special treatment for this channel.

In this form, our expression for A is analogous to the exact expression in Eq. (II A1.4) with two modifications: the additional imaginary term is added to the energy difference, and the sum over the channels includes only the "particle channels" (non-eliminated channels). It is therefore possible to immediately write the R-matrix formula for the eliminated-channel approximation as

$$R_{cc'} = \sum_\lambda \frac{\gamma_{\lambda c}\, \gamma_{\lambda c'}}{E_\lambda - E - i\overline{\Gamma}_{\lambda\gamma}/2}\,\delta_{JJ'} \quad , \tag{II A1.15}$$

where the channel indices c and c' refer only to particle channels, not to the gamma channels. This formula for the R-matrix is the Reich-Moore approximation and is the form which is used in the SAMMY code. See Section II.B.1 for more about this formulation of R-matrix theory.