II.A.2.a. Relating the scattering matrix to the cross sections

The relationship between the scattering matrix U and the cross section σ is also described by many authors; see, for example, [AF71]. Here we provide a summary for the simplest case.

The wave function for a spinless particle far from the scattering source can be written as

$$\psi\left(r,\theta\right)=e^{ikz}+\frac{e^{ikr}}{r}f\left(\theta\right) \quad , \qquad \text{(II A2 a.1)}$$

where f has the form

$$f\left(\theta\right)=\frac{1}{2ik}\sum_{l}\left(2l+1\right)\left[U_l-1\right]P_l\left(\cos\theta\right) \quad . \qquad \text{(II A2 a.2)}$$

The cross section is then given by

$$\frac{d\sigma}{d\Omega}=\left|f\left(\theta\right)\right|^2 \quad . \qquad \text{(II A2 a.3)}$$

For angle-integrated cross sections, the equation found by inserting Eq. (II A2 a.2) into Eq. (II A2 a.3) can be integrated to give

$$\begin{aligned}\sigma = \int & \left[-\frac{1}{2ik}\sum_{l}\left(2l+1\right)\left[U_l^*-1\right]P_l\left(\cos\theta\right)\right] \\ & \times\left[\frac{1}{2ik}\sum_{l'}\left(2l'+1\right)\left[U_{l'}-1\right]P_{l'}\left(\cos\theta\right)\right]d\left(\cos\theta\right)\,d\varphi \\ = & \ \frac{1}{4k^2}\sum_{l\,l'}\left(2l+1\right)\left(2l'+1\right)\left[U_l^*-1\right]\left[U_{l'}-1\right]\int_0^{2\pi}d\varphi\int_{-1}^{1}P_l\left(\cos\theta\right)P_{l'}\left(\cos\theta\right)d\left(\cos\theta\right) \qquad \text{(II A2 a.4)} \\ = & \ \frac{1}{4k^2}\sum_{l\,l'}\left(2l+1\right)\left(2l'+1\right)\left[U_l^*-1\right]\left[U_{l'}-1\right]2\pi\frac{2}{2l+1}\delta_{l\,l'} \\ = & \ \frac{\pi}{k^2}\sum_{l}\left(2l+1\right)\left|U_l-1\right|^2 \quad .\end{aligned}$$

This is analogous to the "standard" scattering theory equation shown in Eq. (II A.1).